ON THE LOMMEL-MAITLAND TRANSFORM IN $\mathfrak{L}_{\nu,r}$-SPACE

Anatoly A. Kilbas[1], Megumi Saigo[2] and Anna Borovco[1]

Abstract

The paper is devoted to study the integral transform

$$\left(\mathbb{J}^{\gamma}_{\eta,\sigma}f\right)(x)=\int_0^\infty (xt)^{1/2}\,\mathbf{J}^{\gamma}_{\eta,\sigma}(xt)f(t)\,dt \quad (x>0)$$

containing the Lommel-Maitland function

$$\mathbf{J}^{\gamma}_{\eta,\sigma}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+\sigma+k)\,\Gamma(1+\sigma+\eta+\gamma k)}\left(\frac{z}{2}\right)^{\eta+2k+2\sigma},$$

with $0<\gamma<1$ and $\sigma,\eta\in\mathbb{C}$ $(\mathrm{Re}(\sigma)>-1,\mathrm{Re}(\eta+\sigma)>-1)$ in the kernel on the space $\mathfrak{L}_{\nu,r}$ of functions f such that

$$\int_0^\infty |t^\nu f(t)|^r\,\frac{dt}{t}<\infty \quad (1\leqq r<\infty,\ \nu\in\mathbb{R}).$$

Mapping properties such as the boundedness, the representation and the range of the transform $\mathbb{J}^{\gamma}_{\eta,\sigma}$ are proved.

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1. Introduction

The paper deals with the integral transform of the form

$$\left(\mathbb{J}^{\gamma}_{\eta,\sigma}f\right)(x)=\int_0^\infty (xt)^{1/2}\,\mathbf{J}^{\gamma}_{\eta,\sigma}(xt)f(t)\,dt \quad (x>0) \tag{1.1}$$

with the kernel function

$$\mathbf{J}^{\gamma}_{\eta,\sigma}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+\sigma+k)\Gamma(1+\sigma+\eta+\gamma k)}\left(\frac{z}{2}\right)^{\eta+2k+2\sigma}, \tag{1.2}$$

where $0<\gamma<1$, $\sigma,\eta\in\mathbb{C}$ with $\mathrm{Re}(\sigma)>-1, \mathrm{Re}(\eta+\sigma)>-1$. The function $\mathbf{J}^{\gamma}_{\eta,\sigma}(z)$ is called the Lommel-Maitland function. The transform (1.1) is clearly defined for continuous functions $f\in C_0$ with compact support on $\mathbb{R}_+=(0,\infty)$.

When $\sigma=0$,

$$\mathbf{J}^{\gamma}_{\eta}(z)\equiv\mathbf{J}^{\gamma}_{\eta,0}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(1+\eta+\gamma k)}\left(\frac{z}{2}\right)^{\eta+2k}, \tag{1.3}$$

and (1.1) takes the form

$$\left(\mathbb{J}^{\gamma}_{\eta}f\right)(x)\equiv\left(\mathbb{J}^{\gamma}_{\eta,0}f\right)(x)=\int_0^\infty (xt)^{1/2}\,\mathbf{J}^{\gamma}_{\eta}(xt)f(t)\,dt \quad (x>0). \tag{1.4}$$

We note that the Bessel-Maitland function $J^{\gamma}_{\eta}(z)$ (see [16] and [21]) is expressed via (1.3) by

$$J^{\gamma}_{\eta}(z)=\sum_{k=0}^{\infty}\frac{(-z)^k}{k!\Gamma(1+\eta+\gamma k)}=z^{-\eta/2}\mathbf{J}^{\gamma}_{\eta}(2\sqrt{z}). \tag{1.5}$$

We also mention that for $\gamma=1$ the transform (1.1) is known as the Hardy transform [26, Section 8.4] which for $\sigma=0$ becomes the Hankel transform

$$\left(\mathbb{H}_{\eta}f\right)(x)=\int_0^\infty (xt)^{1/2}\,J_{\eta}(xt)f(t)\,dt \quad (x>0) \tag{1.6}$$

containing the Bessel function of the first kind $J_\eta(z)$ ([8, Section 7.2.1]) in the kernel.

The Lommel-Maitland transform (1.1) is investigated in the spaces $\mathfrak{L}_{\nu,r}$ of complex-valued Lebesgue measurable functions f on $\mathbb{R}_+$ such that $\|f\|_{\nu,r}<\infty$, where

$$\|f\|_{\nu,r}=\int_0^\infty |t^\nu f(t)|^r\frac{dt}{t} \quad (1\leqq r<\infty,\ \nu\in\mathbb{R}=(-\infty,\infty)) \tag{1.7}$$

and

$$\|f\|_{\nu,\infty} = \operatorname*{ess\,sup}_{x>0}[x^{\nu}|f(x)|] \quad (\nu \in \mathbb{R}). \tag{1.8}$$

We study the mapping properties such as the boundedness, the representation and the range of the Lommel-Maitland transform $\mathbb{J}^{\gamma}_{\eta,\sigma} f$ in (1.1).

The results obtained make more precise than those given by Betancor [4] who proved the boundedness of the Lommel-Maitland transform $\mathbb{J}^{\gamma}_{\eta,\sigma}$ with real η from $\mathfrak{L}_{\nu,r}$ into $\mathfrak{L}_{1-\nu,s}$ for $1 < r < \infty$, $\nu \in \mathbb{R}$, $\max[1/r, 1-1/r] \leqq \nu < 1$ and $r \leqq s < 1/(1-\nu)$. We also mention that the Lommel-Maitland transform (1.1) was first considered by Pathak [18] - [20], who obtained some elementary properties of this transform, proved an inversion formula, indicated the relation of this transform with the Laplace transform, and applied the results to evaluate a number of infinite integrals involving Meijer's G-function. The particular case of the Lommel-Maitland transform $\mathbb{J}^{\gamma}_{\eta} f$ in (1.4) in the form

$$(h_{\eta;\gamma} f)(x) = 2^{-\eta} \int_0^{\infty} (xt)^{\eta+1/2} J^{\gamma}_{\eta}\left(\frac{x^2 t^2}{4}\right) f(t)\, dt \quad (x > 0), \tag{1.9}$$

where $J^{\gamma}_{\eta}(z)$ is the Bessel-Maitland function (1.5), was considered by Agarwal [1] - [3], who gave the Parseval relation and some other properties of this transform, and proved three inversion formulas in a similar form to (1.4), in terms of a double integral and as a diferential operator of infinite order applying to an infinite integral. Betancor [5] obtained another invesrion formula and proved Abelian theorems for the transform $h_{\eta;\gamma}$.

We also mention some results concerning the integral transforms with the Bessel-Maitland function (1.5) in the kernel. Marichev [16] studied the transform

$$\left(\mathcal{J}^{\gamma}_{\eta} f\right)(x) = \int_0^{\infty} (xt)^{1/2} J^{\gamma}_{\eta}(xt) f(t)\, dt \quad (x > 0) \tag{1.10}$$

in $L_2(\mathbb{R}_+)$. Gupta and Jain [11] - [12] investigated another modification

$$\left(\mathcal{J}^{\gamma}_{\eta;\sigma} f\right)(x) = \int_0^{\infty} (xt)^{\sigma} J^{\gamma}_{\eta}(xt) f(t)\, dt \quad (x > 0) \tag{1.11}$$

in some spaces of generalized functions.

We show that the Lommel-Maitland transform (1.1) is a special case of the so-called $\boldsymbol{H}$-transfrom

$$(\boldsymbol{H} f)(x) = \int_0^{\infty} H^{m,n}_{p,q}\left[xt \,\middle|\, \begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q \beta_q) \end{matrix} \right] f(t)\, dt \tag{1.12}$$

with the H-function as kernel - see, for example, [17, Chapter 2], [21, Section 8.3] and [25, Chapter 1]. This transfrom has the property

$$(\mathfrak{M}\boldsymbol{H}f)(s) = \mathcal{H}(s)(\mathfrak{M}f)(1-s), \tag{1.13}$$

with

$$\mathcal{H}(s) = \mathcal{H}_{p,q}^{m,n}\left[\begin{array}{c} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ (b_1,\beta_1),\cdots,(b_q\beta_q) \end{array}\middle| s\right]$$

$$= \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}$$

$$= \left(\mathfrak{M}H_{p,q}^{m,n}\left[t\middle|\begin{array}{c} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ (b_1,\beta_1),\cdots,(b_q\beta_q) \end{array}\right]\right)(s) \tag{1.14}$$

under certain conditions on the function f. Here m, n, p, q are integers such that $0 \leqq m \leqq q$, $0 \leqq n \leqq p$; $a_i,\ b_j \in \mathbb{C}$; $\alpha_i,\ \beta_j \in \mathbb{R}_+$ $(1 \leqq i \leqq p,\ 1 \leqq j \leqq q)$, and an empty product, if it occurs, is taken to be one. $\mathfrak{M}$ is the Mellin transfrom defined by

$$(\mathfrak{M}f)(s) = \int_0^\infty f(t)\, t^{s-1} dt. \tag{1.15}$$

It shoud be noted that for $f \in \mathfrak{L}_{\nu,r}$ with $1 \leqq r \leqq 2$ the Mellin transform $\mathfrak{M}$ is given by

$$(\mathfrak{M}f)(s) = \int_{-\infty}^{\infty} e^{(\sigma+it)\tau} f(e^\tau)\, d\tau \quad (s = \sigma + it,\ \sigma, t \in \mathbb{R}), \tag{1.16}$$

and if $f \in \mathfrak{L}_{\nu,r} \bigcap \mathfrak{L}_{\nu,1}$ and $\mathrm{Re}(s) = \nu$, then (1.16) coincides with (1.15), see [22].

Mapping properties such as the boundedness, the representation and the range of the $\boldsymbol{H}$-transform (1.12) were proved independently in [9] - [10], [13] - [15] and [6] while the invertibility of (1.12) in $\mathfrak{L}_{\nu,r}$ was given in [24]. In this paper we apply the results in [9] - [10] and [13] - [15] to investigate such properties of the transform $\mathbb{J}_{\eta,\sigma}^{\gamma}$. Section 2 deals with some results from the $\mathfrak{L}_{\nu,r}$-theory of the $\boldsymbol{H}$-transfrom (1.12). The Mellin transform of the Lommel-Maitland function (1.2) is given in Section 3. Section 4 is devoted to $\mathfrak{L}_{\nu,2}$-theory of the transform $\mathbb{J}_{\eta,\sigma}^{\gamma}$. In Section 5 we present the boundedness, the range and the representation of the Lommel-Maitland transform in the space $\mathfrak{L}_{\nu,r}$ for any $r \geqq 1$.

2. Auxiliary Results

In this section we give some results from the theory of the the $\boldsymbol{H}$-transform (1.12) on $\mathfrak{L}_{\nu,r}$-spaces given in [13] and [10]. Following these papers we use the notation

$$\alpha = \begin{cases} \max\left[-\frac{\mathrm{Re}(b_1)}{\beta_1}, \cdots, -\frac{\mathrm{Re}(b_m)}{\beta_m}\right] & (m > 0)\,, \\ -\infty & (m = 0)\,; \end{cases} \tag{2.1}$$

$$\beta = \begin{cases} \min\left[\frac{1-\mathrm{Re}(a_1)}{\alpha_1}, \cdots, \frac{1-\mathrm{Re}\,(a_n)}{\alpha_n}\right] & (n > 0)\,, \\ \infty & (n = 0)\,; \end{cases} \tag{2.2}$$

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j; \tag{2.3}$$

$$a_1^* = \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i, \qquad a_2^* = \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j; \tag{2.4}$$

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i, \tag{2.5}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}, \tag{2.6}$$

$$\delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j}. \tag{2.7}$$

We denote by $\mathcal{E}_{\mathcal{H}}$ the exceptional set of the function $\mathcal{H}\,(s)$ which is the set of real numbers ν such that $\alpha < 1-\nu < \beta$ and $\mathcal{H}\,(s)$ has a zero on the line $\mathrm{Re}\,(s) = 1-\nu$. We denote by $[X, Y]$ the collection of bounded linear operators from a Banach space X into a Banach space Y.

The $\mathfrak{L}_{\nu,2}$-theory of the $\boldsymbol{H}$-transform is given by the following theorems.

THEOREM A. ([13, Theorem 3]) *Let $\alpha < 1-\nu < \beta$ and either $a^* > 0$ or $a^* = 0,\ \Delta\,(1-\nu) + \mathrm{Re}\,(\mu) \leqq 0$. Then the following assertions hold*:

(a) *There is a one-to-one transform $\boldsymbol{H} \in [\mathfrak{L}_{\nu,2}, \mathfrak{L}_{1-\nu,2}]$ so that the relation* (1.13) *holds for $f \in \mathfrak{L}_{\nu,2}$ and* $\mathrm{Re}\,(s) = 1-\nu$. *If $a^* = 0,\ \Delta\,(1-\nu) + \mathrm{Re}\,(\mu) = 0$ and $\nu \notin \mathcal{E}_{\mathcal{H}}$, then the operator $\boldsymbol{H}$ maps $\mathfrak{L}_{\nu,2}$ onto $\mathfrak{L}_{1-\nu,2}$.*

(b) *For $f, g \in \mathfrak{L}_{\nu,2}$ the relation of integration by parts holds*

$$\int_0^\infty f\,(x)\,(\boldsymbol{H}g)\,(x)\,dx = \int_0^\infty (\boldsymbol{H}f)\,(x)\,g\,(x)\,dx. \tag{2.8}$$

(c) *Let $f \in \mathfrak{L}_{\nu,2}$, $\lambda \in \mathbb{C}$ and $h \in \mathbb{R}_+$. If* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$, *then $\boldsymbol{H}f$ is given by*

$$(\boldsymbol{H}f)(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}$$

$$\cdot \int_0^\infty H_{p+1,q+1}^{m,n+1} \left[xt \,\middle|\, \begin{matrix} (-\lambda, h), (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q), (-\lambda - 1, h) \end{matrix} \right] f(t)\, dt. \quad (2.9)$$

When $\mathrm{Re}(\lambda) < (1-\nu)h - 1$,

$$(\boldsymbol{H}f)(x) = -hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}$$

$$\cdot \int_0^\infty H_{p+1,q+1}^{m+1,n} \left[xt \,\middle|\, \begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p), (-\lambda, h) \\ (-\lambda - 1, h), (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix} \right] f(t)\, dt. \quad (2.10)$$

(d) *$\boldsymbol{H}$ is independent on ν in the sense that, for ν_1 and ν_2 satisfying the assumptions and for respective transforms $\boldsymbol{H}_1$ on $\mathfrak{L}_{\nu_1,2}$ and $\boldsymbol{H}_2$ on $\mathfrak{L}_{\nu_2,2}$ given in* (1.13), *$\boldsymbol{H}_1 f = \boldsymbol{H}_2 f$ is valid for $f \in \mathfrak{L}_{\nu_1,2} \bigcap \mathfrak{L}_{\nu_2,2}$.*

THEOREM B. ([13, Theorem 4]) *Let $\alpha < 1 - \nu < \beta$ and either $a^* > 0$ or $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) < -1$. Then for $x \in \mathbb{R}_+$ and $f \in \mathfrak{L}_{\nu,2}$ the transform $(\boldsymbol{H}f)(x)$ is given in* (1.12).

Some of the results above can be extended to the space $\mathfrak{L}_{\nu,r}$ with any $r \in [1, \infty)$ (see [9] - [10] and [13] - [14]). We note only two of these results.

THEOREM C. (10, Theorem 6.1]) *Let $a^* > 0$, $\alpha < 1-\nu < \beta$ and $1 \leqq r \leqq s \leqq \infty$.*

(a) *The $\boldsymbol{H}$-transform* (1.12) *defined on $\mathfrak{L}_{\nu,2}$ can be extended to $\mathfrak{L}_{\nu,r}$ as an element of $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,s}]$. If $\nu \notin \mathcal{E}_{\mathcal{H}}$ or $1 < r \leqq 2$, then $\boldsymbol{H}$ is a one-to-one transform from $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{1-\nu,s}$.*

(b) *If $f \in \mathfrak{L}_{\nu,r}$ and $g \in \mathfrak{L}_{\nu,s'}$ with $1/s + 1/s' = 1$, then the relation* (2.8) *holds.*

The range of the $\boldsymbol{H}$-transform on $\mathfrak{L}_{\nu,r}$ is different in nine cases [10, Sections 6 and 7]. We shall use here only the one when $a_1^* > 0$ and $a_2^* < 0$. For this we need several special operators: the modified Laplace transform $\mathbb{L}_{k,\alpha}$

$$(\mathbb{L}_{k,\alpha} f)(x) = \int_0^\infty (xt)^{-\alpha} \exp\left(-|k|(xt)^{1/k}\right) f(t)\, dt \quad (x > 0) \qquad (2.11)$$

for $k \in \mathbb{R}$ $(k \neq 0)$ and $\alpha \in \mathbb{C}$, the modified Hankel transform $\mathbb{H}_{k,\eta}$

$$(\mathbb{H}_{k,\eta} f)(x) = \int_0^\infty (xt)^{1/k - 1/2} J_\eta\left(|k|(xt)^{1/k}\right) f(t)\, dt \quad (x > 0) \qquad (2.12)$$

for $k \in \mathbb{R}$ $(k \neq 0)$ and $\eta \in \mathbb{C}$ $(\mathrm{Re}(\eta) > -3/2)$ (see [22] and [23, Section 39.2, note 36.4]), and the elementary transforms M_ξ and W_δ of the forms

$$(M_\xi f)(x) = x^\xi f(x) \quad (\xi \in \mathbb{C}), \tag{2.13}$$

$$(W_\delta f)(x) = f\left(\frac{x}{\delta}\right) \quad (\delta > 0). \tag{2.14}$$

THEOREM D. (10, Theorem 7.1]) *Let* $a_1^* > 0$, $a_2^* < 0$, $\alpha < 1-\nu < \beta$ *and* $1 < r < \infty$.

(a) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$ *or* $1 < r \leqq 2$, *then* $\boldsymbol{H}$ *is a one-to-one transform on* $\mathfrak{L}_{\nu,r}$.

(b) *Let* $\omega, \xi, \zeta \in \mathbb{C}$ *be chosen as*

$$\omega = a^*\xi - \mu - \frac{1}{2}, \tag{2.15}$$

$$a^*\mathrm{Re}(\xi) \geqq \gamma(r) + 2a_2^*(\nu-1) + \mathrm{Re}(\mu), \tag{2.16}$$

$$\mathrm{Re}(\xi) > \nu - 1 \tag{2.17}$$

$$\mathrm{Re}(\zeta) < 1 - \nu, \tag{2.18}$$

where μ *is given in* (2.6) *and* $\gamma(r) = \max[1/r, 1-1/r]$. *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}(\mathfrak{L}_{\nu,r}) = \left(W_\delta M_{1/2+\omega/(2a_2^*)} \mathbb{H}_{-2a_2^*, 2a_2^*\zeta+\omega-1} \mathbb{L}_{-a_2^*, 1/2+\xi-\omega/(2a_2^*)}\right) \left(\mathfrak{L}_{3/2-\nu+\mathrm{Re}(\omega)/(2a_2^*), r}\right), \tag{2.19}$$

where δ *is given in* (2.7). *When* $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (2.19).

3. Mellin Transform of the Lommel-Maitland Function

The $\mathfrak{L}_{\nu,r}$-theory of the Lommel-Maitland transform (1.1) is based on the Mellin transform of the Lommel-Maitland function (1.2). Such a transform is given by the following assertion.

LEMMA 1. *Let* $0 < \gamma < 1$ *and* $\sigma, \eta \in \mathbb{C}$ *satisfy the relations*

$$\mathrm{Re}(\sigma) > -1, \quad \mathrm{Re}(\eta + \sigma) > -1, \tag{3.1}$$

then for $s \in \mathbb{C}$ *with*

$$-\frac{1}{2} < \mathrm{Re}(s) + \mathrm{Re}(\eta + 2\sigma) < \frac{3}{2}, \tag{3.2}$$

there holds the equality

$$\left(\mathfrak{M}[x^{1/2}\mathbf{J}^{\gamma}_{\eta,\sigma}(x)]\right)(s) = 2^{-\eta-2\sigma}\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}+\eta+2\sigma+s\right)\Gamma\left(\frac{3}{4}-\frac{\eta}{2}-\sigma-\frac{s}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{\eta}{2}+\sigma+\frac{s}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\eta}{2}-\frac{s}{2}\right)}$$

$$\cdot\frac{1}{\Gamma\left(1+\eta+\sigma-\gamma\left[\frac{1}{4}+\frac{\eta}{2}+\sigma\right]-\frac{\gamma s}{2}\right)}. \tag{3.3}$$

When $\sigma = 0$, *the function* $\mathbf{J}^{\gamma}_{\eta,\sigma}(x)$ *is replaced by* $\mathbf{J}^{\gamma}_{\eta}(x)$, *and the right inequality of* (3.2) *is excluded.*

Proof. It is known (see Betancor [4]) that

$$\left(\mathfrak{M}[x^{1/2}\mathbf{J}^{(\gamma)}_{\eta,\sigma}(x)]\right)(s)$$

$$= 2^{s-1/2}\frac{\Gamma\left(\frac{1}{4}+\frac{\eta}{2}+\sigma+\frac{s}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\eta}{2}-\sigma-\frac{s}{2}\right)}{\Gamma\left(\frac{3}{4}-\frac{\eta}{2}-\frac{s}{2}\right)\Gamma\left(1+\eta+\sigma-\gamma\left[\frac{1}{4}+\sigma+\frac{\eta}{2}\right]-\frac{\gamma s}{2}\right)}. \tag{3.4}$$

Multiplying the numerator and the denominator of the right hand side of (3.4) by $\Gamma(3/4+\eta/2+\sigma+s/2)$ and using the Legendre duplication formula (see [7, 1.2(15)])

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{3.5}$$

we easily have the relation (3.3).

If $\sigma \neq 0$, then according to (3.3) and (1.14), $\left(\mathfrak{M}[x^{1/2}\mathbb{J}^{\gamma}_{\eta,\sigma}(x)]\right)(s)$ is represented by the function $\mathcal{H}$ of the form

$$\left(\mathfrak{M}[x^{1/2}\mathbf{J}^{\gamma}_{\eta,\sigma}(x)]\right)(s) = 2^{-\eta-2\sigma}\sqrt{\pi}$$

$$\cdot\,\mathcal{H}^{1,1}_{2,3}\left[\begin{array}{c}\left(\frac{\eta}{2}+\sigma+\frac{1}{4},\frac{1}{2}\right),\left(\frac{\eta}{2}+\sigma+\frac{3}{4},\frac{1}{2}\right)\\ \left(\eta+2\sigma+\frac{1}{2},1\right),\left(\frac{\eta}{2}+\frac{1}{4},\frac{1}{2}\right),\left(\gamma\left[\frac{\eta}{2}+\sigma+\frac{1}{4}\right]-\eta-\sigma,\frac{\gamma}{2}\right)\end{array}\middle| s\right]. \tag{3.6}$$

In particular, when $\sigma = 0$, (3.3), (1.4), (1.14) and the property of lowering the order of the H-function (see, for example [21, 8.3.26]) imply

$$\left(\mathfrak{M}[x^{1/2}\mathbf{J}^{\gamma}_{\eta}(x)]\right)(s) = 2^{-\eta}\sqrt{\pi}\mathcal{H}^{1,0}_{1,2}\left[\begin{array}{c}\left(\frac{\eta}{2}+\frac{3}{4},\frac{1}{2}\right)\\ \left(\eta+\frac{1}{2},1\right),\left(\gamma\left[\frac{\eta}{2}+\frac{1}{4}\right]-\eta,\frac{\gamma}{2}\right)\end{array}\middle| s\right]. \tag{3.7}$$

4. $\mathfrak{L}_{\nu,2}$-theory of the Lommel-Maitland Transform $\mathbb{J}^{\gamma}_{\eta,\sigma}$

In view of (3.6), the transform $\mathbb{J}^{\gamma}_{\eta,\sigma}$ is regarded as a framwork of $\boldsymbol{H}$-transform (1.12):

$$\left(\mathbb{J}^{\gamma}_{\eta,\sigma} f\right)(x) = 2^{-\eta-2\sigma}\sqrt{\pi}$$

$$\cdot \int_0^{\infty} H^{1,1}_{2,3}\left[xt \left| \begin{matrix} \left(\frac{\eta}{2}+\sigma+\frac{1}{4}, \frac{1}{2}\right), & \left(\frac{\eta}{2}+\sigma+\frac{3}{4}, \frac{1}{2}\right) \\ \left(\eta+2\sigma+\frac{1}{2}, 1\right), & \left(\frac{\eta}{2}+\frac{1}{4}, \frac{1}{2}\right), \left(\gamma\left[\frac{\eta}{2}+\sigma+\frac{1}{4}\right]-\sigma-\eta, \frac{\gamma}{2}\right) \end{matrix} \right.\right] f(t)\,dt. \quad (4.1)$$

In particular, the transform $\mathbb{J}^{\gamma}_{\eta}$ in (1.4) is the $\boldsymbol{H}$-transform of the form, as $\sigma = 0$ in (4.1),

$$\left(\mathbb{J}^{\gamma}_{\eta} f\right)(x) = 2^{-\eta}\sqrt{\pi}\int_0^{\infty} H^{1,0}_{1,2}\left[xt \left| \begin{matrix} \left(\frac{\eta}{2}+\frac{3}{4}, \frac{1}{2}\right) \\ \left(\eta+\frac{1}{2}, 1\right), \left(\gamma\left[\frac{\eta}{2}+\frac{1}{4}\right]-\eta, \frac{\gamma}{2}\right) \end{matrix} \right.\right] f(t)\,dt. \quad (4.2)$$

Now we apply the results in Section 2 to charactirize $\mathfrak{L}_{\nu,r}$-properties of the Lommel-Maitland transform (1.1). First we consider such a property for $r = 2$. According to (3.6) and (2.1) - (2.6) we have

$$\begin{gathered} \alpha = -\mathrm{Re}\,(\eta+2\sigma) - \frac{1}{2}, \qquad \beta = \frac{3}{2} - \mathrm{Re}\,(\eta+2\sigma), \\ a^* = \frac{1-\gamma}{2}, \qquad \Delta = \frac{1+\gamma}{2}, \\ \delta = \sqrt{2}\left(\frac{\gamma}{2}\right)^{\gamma/2}, \qquad \mu = (\gamma-1)\left(\sigma+\frac{\eta}{2}+\frac{1}{4}\right) - \frac{1}{2}, \\ a_1^* = \frac{1}{2}, \qquad a_2^* = -\frac{\gamma}{2} \end{gathered} \quad (4.3)$$

for the transform (4.1). For the transform (4.2) the same can be applied with $\sigma = 0$ except $\beta = +\infty$.

Let $\mathcal{E}_{\mathcal{H}}$ be the exceptional set of the function $\mathcal{H}^{1,1}_{2,3}$ in (3.6). By virtue of (3.3) and (4.3), ν is not in the exceptional set $\mathcal{E}_{\mathcal{H}}$, if

$$\begin{gathered}\nu \geqq \frac{3}{2}+\operatorname{Re}(\eta+2\sigma) \quad \text{or} \quad \nu \leqq -\frac{1}{2}+\operatorname{Re}(\eta+2\sigma), \quad \text{and} \\ s \neq -\eta-2\sigma-\frac{3}{2}-2k \quad (k=0,1,2,\cdots), \\ s \neq \frac{3}{2}-\eta+2m \quad (m=0,1,2,\cdots), \\ s \neq -\left(\eta+2\sigma+\frac{1}{2}\right)+\frac{2}{\gamma}(1+\eta+\sigma+n) \quad (n=0,1,2,\cdots)\end{gathered} \tag{4.4}$$

for $\operatorname{Re}(s)=1-\nu$.

When $\sigma=0$, it follows that $\nu \notin \mathcal{E}_{\mathcal{H}}$, for $\mathcal{H}$ being the function $\mathcal{H}_{1,2}^{1,0}$ in (3.7), if

$$\begin{gathered}\nu \geqq \frac{3}{2}+\operatorname{Re}(\eta), \quad s \neq -\eta-\frac{3}{2}-2k \quad (k=0,1,2,\cdots), \\ s \neq -\left(\eta+\frac{1}{2}\right)+\frac{2}{\gamma}(1+\eta+n) \quad (n=0,1,2,\cdots)\end{gathered} \tag{4.5}$$

for $\operatorname{Re}(s)=1-\nu$.

Using (4.1), (4.3) and (4.4) and applying Theorems A and B, we obtain the following result for the Lommel-Maitland transform (1.1) in the space $\mathfrak{L}_{\nu,2}$.

THEOREM 1. *Let $\sigma \in \mathbb{C}$ $(\sigma \neq 0)$, $\eta \in \mathbb{C}$ and $\nu \in \mathbb{R}$ be such that*

$$\begin{gathered}\operatorname{Re}(\sigma)>-1, \quad \operatorname{Re}(\eta+\sigma)>-1, \\ \operatorname{Re}(\eta+2\sigma)-\frac{1}{2}<\nu<\operatorname{Re}(\eta+2\sigma)+\frac{3}{2}\end{gathered} \tag{4.6}$$

and let $0<\gamma<1$.

(i) *There is a one-to-one transform $\mathbb{J}_{\eta,\sigma}^{\gamma} \in [\mathfrak{L}_{\nu,2}, \mathfrak{L}_{1-\nu,2}]$ so that the relation*

$$\begin{aligned}\left(\mathfrak{M}\mathbb{J}_{\eta,\sigma}^{\gamma} f\right)(s) &= 2^{-\eta-2\sigma}\pi^{1/2}\frac{\Gamma\left(\eta+2\sigma+\frac{1}{2}+s\right)\Gamma\left(\frac{9}{4}-\frac{\eta}{2}-\sigma-\frac{s}{2}\right)}{\Gamma\left(\frac{3}{4}+\sigma+\frac{\eta}{2}+\frac{s}{2}\right)\Gamma\left(\frac{9}{4}-\frac{\eta}{2}-\frac{s}{2}\right)} \\ &\quad \cdot \frac{1}{\Gamma\left(1+\eta+\sigma-\gamma\left[\frac{1}{4}+\sigma+\frac{\eta}{2}\right]-\frac{\gamma s}{2}\right)}(\mathfrak{M}f)(1-s)\end{aligned} \tag{4.7}$$

holds for $\operatorname{Re}(s)=1-\nu$ and $f \in \mathfrak{L}_{\nu,2}$. If the conditions in (4.4) are fulfiled, then the operator $\mathbb{J}_{\eta,\sigma}^{\gamma}$ maps $\mathfrak{L}_{\nu,2}$ onto $\mathfrak{L}_{1-\nu,2}$.

(ii) *The relation*

$$\int_0^\infty \left(\mathbb{J}_{\eta,\sigma}^{\gamma} f\right)(x)\, g(x)\, dx=\int_0^\infty f(x)\left(\mathbb{J}_{\eta,\sigma}^{\gamma} g\right)(x)\, dx. \tag{4.8}$$

holds for $f, g \in \mathfrak{L}_{\nu,2}$.

(iii) *Let* $\lambda \in \mathbb{C}$ *and* $h \in \mathbb{R}_+$. *When* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$, $\mathbb{J}^{\gamma}_{\eta,\sigma} f$ *is given by*

$$\left(\mathbb{J}^{\gamma}_{\eta,\sigma} f\right)(x) = 2^{-\eta-2\sigma}\sqrt{\pi} h x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \cdot \int_0^{\infty} H^{2,1}_{3,4}\left[xt \left| \begin{matrix} (-\lambda, h), \left(\frac{\eta}{2}+\sigma+\frac{1}{4}, \frac{1}{2}\right), \left(\frac{\eta}{2}+\sigma+\frac{3}{4}, \frac{1}{2}\right) \\ \left(\eta+2\sigma+\frac{1}{2}, 1\right), \left(\frac{\eta}{2}+\frac{1}{4}, \frac{1}{2}\right), \left(\gamma\left[\frac{\eta}{2}+\sigma+\frac{1}{4}\right]-\eta-\sigma, \frac{\gamma}{2}\right), (-\lambda-1, h) \end{matrix} \right.\right] f(t)\, dt \quad (x > 0). \tag{4.9}$$

When $\mathrm{Re}(\lambda) < (1-\nu)h - 1$,

$$\left(\mathbb{J}^{\gamma}_{\eta,\sigma} f\right)(x) = -2^{-\eta-2\sigma}\sqrt{\pi} h x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \cdot \int_0^{\infty} H^{1,2}_{3,4}\left[xt \left| \begin{matrix} \left(\frac{\eta}{2}+\sigma+\frac{1}{4}, \frac{1}{2}\right), \left(\frac{\eta}{2}+\sigma+\frac{3}{4}, \frac{1}{2}\right), (-\lambda, h) \\ (-\lambda-1, h), \left(\eta+2\sigma+\frac{1}{2}, 1\right), \left(\frac{\eta}{2}+\frac{1}{4}, \frac{1}{2}\right), \left(\gamma\left[\frac{\eta}{2}+\sigma+\frac{1}{4}\right]-\eta-\sigma, \frac{\gamma}{2}\right) \end{matrix} \right.\right] f(t)\, dt \quad (x > 0). \tag{4.10}$$

(iv) $\mathbb{J}^{\gamma}_{\eta,\sigma}$ *is independent of* ν *in the sense that, for* ν_1 *and* ν_2 *satisfying the assumptions and for respective transforms* $\mathbb{J}^{(\gamma)}_{\eta,\sigma;1}$ *on* $\mathfrak{L}_{\nu_1,2}$ *and* $\mathbb{J}^{\gamma}_{\eta,\sigma;2}$ *on* $\mathfrak{L}_{\nu_2,2}$ *given in* (4.7), *there holds* $\mathbb{J}^{\gamma}_{\eta,\sigma;1} f = \mathbb{J}^{\gamma}_{\eta,\sigma;2} f$ *for* $f \in \mathfrak{L}_{\nu_1,2} \bigcap \mathfrak{L}_{\nu_2,2}$.

(v) $\mathbb{J}^{\gamma}_{\eta,\sigma} f$ *is given by* (4.1).

In particular, when $\sigma = 0$, we have the following corollay for the transform $\mathbb{J}^{\gamma}_{\eta}$.

COROLLARY 1. *Let* $\eta \in \mathbb{C}$ *and* $\nu \in \mathbb{R}$ *be such that* $\mathrm{Re}(\eta) > -1$ *and* $\nu < \mathrm{Re}(\eta) + 3/2$.

(i) There is a one-to-one transform $\mathbb{J}^{\gamma}_{\eta} \in [\mathfrak{L}_{\nu,2}, \mathfrak{L}_{1-\nu,2}]$ so that the relation

$$\left(\mathfrak{M}\mathbb{J}^{\gamma}_{\eta} f\right)(s)$$

$$= 2^{-\eta}\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}+\eta+s\right)}{\Gamma\left(\frac{3}{4}+\frac{\eta}{2}+\frac{s}{2}\right)\Gamma\left(1+\eta-\gamma\left[\frac{1}{4}+\frac{\eta}{2}\right]-\frac{\gamma s}{2}\right)}(\mathfrak{M}f)(1-s) \quad (4.11)$$

holds for $\mathrm{Re}\,(s) = 1-\nu$ and $f \in \mathfrak{L}_{\nu,2}$. If the conditions in (4.5) are fulfiled, then the operator $\mathbb{J}_\eta^\gamma$ maps $\mathfrak{L}_{\nu,2}$ onto $\mathfrak{L}_{1-\nu,2}$.

(ii) The relation

$$\int_0^\infty \left(\mathbb{J}_\eta^\gamma f\right)(x)\, g(x)\, dx = \int_0^\infty f(x)\left(\mathbb{J}_\eta^\gamma g\right)(x)\, dx. \quad (4.12)$$

holds for $f, g \in \mathfrak{L}_{\nu,2}$.

(iii) Let $\lambda \in \mathbb{C}$ and $h \in \mathbb{R}_+$. When $\mathrm{Re}\,(\lambda) > (1-\nu)\,h - 1$, $\mathbb{J}_\eta^\gamma f$ is given by

$$\left(\mathbb{J}_\eta^\gamma f\right)(x) = 2^{-\eta}\sqrt{\pi} h x^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}$$

$$\cdot \int_0^\infty H_{2,3}^{1,1}\left[xt\,\middle|\,\begin{matrix}(-\lambda,h),\left(\frac{\eta}{2}+\frac{3}{4},\frac{1}{2}\right)\\ \left(\eta+\frac{1}{2},1\right),\left(\gamma\left[\frac{\eta}{2}+\frac{1}{4}\right]-\eta,\frac{\gamma}{2}\right),(-\lambda-1,h)\end{matrix}\right] f(t)\, dt \quad (4.13)$$

$$(x > 0).$$

When $\mathrm{Re}\,(\lambda) < (1-\nu)\,h - 1$,

$$\left(\mathbb{J}_\eta^\gamma f\right)(x) = -2^{-\eta}\sqrt{\pi} h x^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}$$

$$\cdot \int_0^\infty H_{2,3}^{2,0}\left[xt\,\middle|\,\begin{matrix}\left(\frac{\eta}{2}+\frac{3}{4},\frac{1}{2}\right),(-\lambda,h)\\ (-\lambda-1,h),\left(\eta+\frac{1}{2},1\right),\left(\gamma\left[\frac{\eta}{2}+\frac{1}{4}\right]-\eta-\sigma,\frac{\gamma}{2}\right)\end{matrix}\right] f(t)\, dt \quad (4.14)$$

$$(x > 0).$$

(iv) $\mathbb{J}_{\eta,\sigma}^\gamma$ is independent of ν in the sense that, for ν_1 and ν_2 satisfying $\nu_i < \mathrm{Re}\,(\eta) + 3/2$ $(i = 1, 2)$ and for respective transforms $\mathbb{J}_{\eta;1}^{(\gamma)}$ on $\mathfrak{L}_{\nu_1,2}$ and $\mathbb{J}_{\eta;2}^\gamma$ on $\mathfrak{L}_{\nu_2,2}$ given in (4.11), there holds $\mathbb{J}_{\eta;1}^\gamma f = \mathbb{J}_{\eta;2}^\gamma f$ for $f \in \mathfrak{L}_{\nu_1,2} \bigcap \mathfrak{L}_{\nu_2,2}$.

(v) $\mathbb{J}_\eta^\gamma f$ is given by (4.2).

5. Boundednes and Range of $\mathbb{J}_{\eta,\sigma}^\gamma$ in $\mathfrak{L}_{\nu,r}$

Here we present the results characterizing the boundedeness and the range of the Lommel-Maitland transform (1.1) in the space $\mathfrak{L}_{\nu,r}$. By virtue of (4.1), (4.3) and (4.4), Theorems C and D(a) yield the boundedness of the transform $\mathbb{J}_{\eta,\sigma}^\gamma$.

THEOREM 2. *Let $\sigma \in \mathbb{C}$ $(\sigma \neq 0)$, $\eta \in \mathbb{C}$ and $\nu \in \mathbb{R}$ be such that the conditions in* (4.6) *are satisfied.*

(a) *Let $1 \leqq r \leqq s \leqq \infty$, then the transform $\mathbb{J}^{\gamma}_{\eta,\sigma}$* (1.1) *defined on $\mathfrak{L}_{\nu,2}$ can be extended to $\mathfrak{L}_{\nu,r}$ as an element of $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,s}]$. If $1 < r \leqq 2$, or if $1 < r < \infty$ and the conditions in* (4.4) *are satisfied, then $\mathbb{J}^{\gamma}_{\eta,\sigma}$ is a one-to-one transform from $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{1-\nu,s}$.*

(b) *Let $1 \leqq r \leqq s \leqq \infty$, $f \in \mathfrak{L}_{\nu,r}$ and $g \in \mathfrak{L}_{\nu,s'}$ with $1/s + 1/s' = 1$, then the relation* (4.8) *holds.*

In particular, when $\sigma = 0$ we obtain the boundedness of the transform $\mathbb{J}^{\gamma}_{\eta}$ (1.4) on $\mathfrak{L}_{\nu,r}$.

COROLLARY 2. *Let $\eta \in \mathbb{C}$ and $\nu \in \mathbb{R}$ be such that be such that $\mathrm{Re}(\eta) > -1$ and $\nu < \mathrm{Re}(\eta) + 3/2$.*

(a) *Let $1 \leqq r \leqq s \leqq \infty$, then the transform $\mathbb{J}^{\gamma}_{\eta}$ defined on $\mathfrak{L}_{\nu,2}$ can be extended to $\mathfrak{L}_{\nu,r}$ as an element of $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,s}]$. If $1 < r \leqq 2$, or if $1 < r < \infty$ and the conditions in* (4.5) *are satisfied, then $\mathbb{J}^{\gamma}_{\eta}$ is a one-to-one transform from $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{1-\nu,s}$.*

(b) *Let $1 \leqq r \leqq s \leqq \infty$, $f \in \mathfrak{L}_{\nu,r}$ and $g \in \mathfrak{L}_{\nu,s'}$ with $1/s + 1/s' = 1$, then the relation* (4.12) *holds.*

Theorem D(b) gives the range of the Lommel-Maitland transform $\mathbb{J}^{\gamma}_{\eta,\sigma}$ in the space $\mathfrak{L}_{\nu,r}$.

THEOREM 3. *Let $\sigma \in \mathbb{C}$ $(\sigma \neq 0)$, $\eta \in \mathbb{C}$ and $\nu \in \mathbb{R}$ be such that the conditions in* (4.6) *are satisfied. Let $1 < r < \infty$,*

$$\omega = (1-\gamma)\left(\frac{\xi + \eta}{2} + \sigma + \frac{1}{4}\right), \tag{5.1}$$

where ξ is chosen as

$$\mathrm{Re}(\xi) \geqq \frac{1}{1-\gamma}\left[2\gamma(r) + 2\gamma(1-\nu) - 1\right] - \mathrm{Re}(2\sigma + \eta) - \frac{1}{2}, \quad \mathrm{Re}(\xi) > \nu - 1, \tag{5.2}$$

and let ζ be $\mathrm{Re}(\zeta) < 1 - \nu$. If the conditions in (4.4) *are satisfied, then*

$$\mathbb{J}^{\gamma}_{\eta,\sigma}(\mathfrak{L}_{\nu,r}) = \left(W_{\delta} M_{1/2-\omega/\gamma} \mathbb{H}_{\gamma,\omega-\gamma\zeta-1} \mathbb{L}_{\gamma/2,\xi+1/2+\omega/\gamma}\right)\left(\mathfrak{L}_{3/2-\nu-\mathrm{Re}(\omega/\gamma),r}\right), \tag{5.3}$$

where δ is given in (4.3). *When some condition in* (4.4) *is not satisfied, then $\mathbb{J}^{\gamma}_{\eta,\sigma}(\mathfrak{L}_{\nu,r})$ is a subset of the right hand side of* (5.3).

In particular, when $\sigma = 0$ we obtain:

COROLLARY 3. *Let $\eta \in \mathbb{C}$ and $\nu \in \mathbb{R}$ be such that $\mathrm{Re}(\eta) > -1$ and $\nu < \mathrm{Re}(\eta) + 3/2$. Let $1 < r < \infty$, and let ω be given in* (5.1) *with $\sigma = 0$ and let ξ and ζ be in* (5.2) *with $\sigma = 0$. If the conditions in* (4.5) *are satisfied, then*

$$\mathbb{J}^{\gamma}_{\eta}(\mathfrak{L}_{\nu,r}) = \left(W_{\delta} M_{1/2-\omega/\gamma} \mathbb{H}_{\gamma,\omega-\gamma\zeta-1} \mathbb{L}_{\gamma/2,\xi+1/2+\omega/\gamma}\right)\left(\mathfrak{L}_{3/2-\nu-\mathrm{Re}(\omega/\gamma),r}\right). \tag{5.4}$$

When some condition in (4.5) *is not satisfied, then* $\mathbb{J}_{\eta}^{\gamma}(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.4).

REMARK 1. The Lommel-Maitland transform $\mathbb{J}_{\eta,\sigma}^{\gamma}$ for real η in the spaces $\mathfrak{L}_{\nu,r}$ with $1 < r < \infty$, $\nu \in \mathbb{R}$ and $\gamma(r) \leqq \nu < 1$ was considered by Betancor [4]. In the paper the boundedness of $\mathbb{J}_{\eta,\sigma}^{\gamma}$ from $\mathfrak{L}_{\nu,r}$ into $\mathfrak{L}_{1-\nu,s}$ is established, under the assumptions $0 < \gamma < 1$ and

$$\sigma > -1, \quad \eta + \sigma > -1, \quad 1 + \max\left[1, \eta + 2\sigma - \frac{9}{2}\right] < \nu < \eta + 2\sigma + \frac{3}{2}, \tag{5.5}$$

which are more hard than the conditions (4.6) of Theorem 2(a) for real η.

REMARK 2. Betancor [4] gave conditions for the characterizing the range of the Lommel-Maitland transform (1.1) in terms of the Fourier cosine-transform

$$(\mathbb{F}_c f)(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \cos(xt) f(t)\, dt \quad (x > 0) \tag{5.6}$$

in the forms

$$\mathbb{J}_{\eta,\sigma}^{\gamma}(\mathfrak{L}_{\nu,r}) \subseteq \mathbb{F}_c(\mathfrak{L}_{\nu,r}) \tag{5.7}$$

and

$$\mathbb{J}_{\eta,\sigma}^{\gamma}(\mathfrak{L}_{\nu,r}) = \mathbb{F}_c(\mathfrak{L}_{\nu,r}). \tag{5.8}$$

He also obtained the conditions for the imbedding

$$\mathbb{J}_{\eta_1,\sigma_1}^{\gamma_1}(\mathfrak{L}_{\nu,r}) \subseteq \mathbb{J}_{\eta_2,\sigma_2}^{\gamma_2}(\mathfrak{L}_{\nu,r}) \tag{5.9}$$

for different γ_i, η_i and σ_i $(i = 1, 2)$.

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[1] *Department of Mathematics and Mechanics*
Belarusian State University
220050 Minsk, BELARUS
e-mail: kilbas@mmf.bsu.unibel.by

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[2] *Department of Applied Mathematics*
Fukuoka University
Fukuoka 814-0180, JAPAN
e-mail: msaigo@fukuoka-u.ac.jp